Slides Mathematical Finance I Master QFin

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Admin

- Examination: Midterm Exam (30%) on 6.6., Exercise Series (20%), Final exam (50%)
- Overall score of 50% and at least 45% in final are necessary for passing
- Exercises in groups of 3, return by tuesday morning via Canvas,
- Mandatory and additional voluntary assignments are discussed in tutorium (by Ana Rakovic) (Tue 9.00 - 9.55)
- Literature:
  - Slides and lecture notes continuous-time finance (mandatory)
  - [Shreve, 2004], [Björk, 2004] (supplementary)

Overview

Topics in Modern Mathematical Finance.

- Pricing and hedging of derivatives. (Financial derivatives are securities whose value is derived from the price of an underlying security
- Portfolio optimisation. Construction of a portfolio that is optimised under the aspect of risk versus performance,

Closely related disciplines

- Risk management. Risk management deals with measuring and managing financial risk. Close relations with mathematical finance but emphasis more on statistics and examination of large portfolios.
- Statistics of financial markets and financial econometrics. (Analysis of financial data, statistical estimation of models)
- Insurance mathematics.

Mathematical techniques used in mathematical finance are founded in stochastics (probability theory, stochastic processes, statistics), but also optimization, calculus and numerics.

Overview

Stochastic processes in discrete time

Consider some $(\Omega, \mathcal{F}, P)$ with filtration $\mathbb{F} = (\mathcal{F}_n)_{n=0,1,\ldots,N}$. Then a stochastic process X is a sequence of rvs $X_n(\omega)$, $n = 0, 1, \ldots, N$.

Definition. Consider some stochastic process X.

- For ω fixed the mapping $n \mapsto X_n(\omega)$ is called sample path or trajectory of the process.
- X is called adapted with respect to $\mathbb{F}$ if X_n is $\mathcal{F}_n$ measurable for all n.
- X is called predictable with respect to $\mathbb{F}$ if X_n is $\mathcal{F}_{n-1}$ measurable for all $n \geq 1$.

Martingales

Definition 1

An adapted process $(X_n)_n$ with $E(|X_n|) < \infty$ is called a martingale, if for all $n \in \mathbb{N}$ $E(X_{n+1}|\mathcal{F}_n) = X_n$.

Comments.

- Martingale property is equivalent to $E(X_{n+1} - X_n|\mathcal{F}_n) = 0$ for all n.
- Martingales are constant in expectation. In fact we have

$$E(M_n) = E(E(M_n|\mathcal{F}_{n-1})) = E(M_{n-1}) = \cdots = E(M_0)\,.$$

Examples for martingales

1) Sums of independent rvs (random walk): Given a sequence $Y_1, Y_2, ...$ of iid rvs variables with $E(Y_1) = 0$. Define $\mathcal{F}_n := \sigma(Y_1, ..., Y_n)$. Then $S_n := \sum_{i=1}^n Y_i$ is a martingale. In fact we have, as Y_{n+1} is independent of $\mathcal{F}_n$

$$E(S_{n+1} - S_n | \mathcal{F}_n) = E(Y_{n+1} | \mathcal{F}_n) = E(Y_{n+1}) = 0\,.$$

2) Successive predictions: Consider a random variable V and a filtration $\{\mathcal{F}_n\}$. Define $M_n = E(V|\mathcal{F}_n)$. Then M is a martingale, as $E(M_{n+1}|\mathcal{F}_n) = E(E(V|\mathcal{F}_{n+1})|\mathcal{F}_n) = E(V|\mathcal{F}_n) = M_n$. (on finite-horizon models every martingale is of this form for some V).

Sub- and Supermartingales

Definition 2

Consider an an adapted process $X = (X_n)_n$ with $E(|X_n|) < \infty$

- The process is called a supermartingale if for all $n \in \mathbb{N}$ $E(X_{n+1}|\mathcal{F}_n) \leq X_n$.
- The process is called a submartingale if for all $n \in \mathbb{N}$ $E(X_{n+1}|\mathcal{F}_n) \geq X_n$.

- Note that X is a martingale iff it is simultaneously a sub- and a supermartingale.
- A super (sub-) martingale is decreasing (increasing) on average.

Doob decomposition

Theorem 3 (Doob decomposition.)

Consider an adapted stochastic process $X = (X_n)_{n=0,1,2,\ldots}$ on $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and assume that $E(X_n) < \infty$ for all n. Then there is a unique decomposition

$$X_n = M_n + A_n, n \in \mathbb{N} \text{ such that} \tag{1}$$

$M = (M_n)_{n\in\mathbb{N}}$ is a martingale and $A = (A_n)_{n\in\mathbb{N}}$ is a predictable process with $A_0 = 0$. The process A is defined recursively by

$$A_0 = 0, \;\; A_{n+1} = A_n + E(X_{n+1} - X_n \mid \mathcal{F}_n). \tag{2}$$

Corollary. The process $(X_n)_{n\in\mathbb{N}}$ is a supermartingal (submartingal) iff the predictable process $(A_n)_{n\in\mathbb{N}}$ from the Doob decomposition is decreasing (increasing).

Doob decomposition ctd

Proof. a) Uniqueness. Let $X = M + A$ be a decomposition as in the theorem. Then we get

$$E(X_{n+1} - X_n \mid \mathcal{F}_n) = E(M_{n+1} - M_n \mid \mathcal{F}_n) + E(A_{n+1} - A_n \mid \mathcal{F}_n) = A_{n+1} - A_n$$

hence for given A_n, A_{n+1} is uniquely determined and hence also M_{n+1}.
b) Existence. The candidate for A_n is predictable. Moreover, we have

$$E(M_{n+1} - M_n \,|\mathcal{F}_n\,) = E(X_{n+1} - X_n \,|\mathcal{F}_n\,) - \underbrace{E(A_{n+1} - A_n \,|\mathcal{F}_n)}_{=E(X_{n+1} - X_n \,|\mathcal{F}_n)} = 0$$

Stopping times

Consider some $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Definition 4

A random variable τ with values in $N \cup \{\infty\}$ is called a *stopping time* (wrt. $(\mathcal{F}_n)_{n\in\mathbb{N}}$), if

$$\{\tau \leq n\} = \{\omega \in \Omega : \tau(\omega) \leq n\} \in \mathcal{F}_n \qquad \forall\, n \leq \infty.$$

Comments.

- τ is a stopping time iff $\{\tau = n\} \in \mathcal{F}_n$ for all n.
- Definition formalizes the idea that the decision to "stop" (e.g. to start an investment project) is based on only past and present information.
- Typical examples: Let X be an adapted stochastic process and $A \subset \mathbb{R}$ a Borel measurable set. Then the first hitting time of A, defined by $\tau_A = \inf\{n \in \mathbb{N} \colon X_n \in A\}$ is a stopping time

The σ-field $\mathcal{F}_\tau$

We need a formal notion for events that can be observed up to some random time τ.

Definition 5

Given a stopping time τ we define the σ field of all events that can be observed up to time τ by

$$\mathcal{F}_\tau := \Big\{ A \in \mathcal{F} : A \cap \{\tau \leq n\} \in \mathcal{F}_n \text{ for all } n \Big\}$$

Proposition. We have the following properties of $\mathcal{F}_\tau$

1. $\mathcal{F}_\tau$ is a σ-field
2. For two stopping times τ, σ with t $\sigma \leq \tau$ one has $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.
3. For an adapted process $X = (X_n)_{n \in \mathbb{N}}$ and a finite stopping time τ the rv $X_\tau = \sum_{n=1}^{\infty} 1_{\{\tau = n\}} X_n$ is $\mathcal{F}_\tau$-measurable.

Optional sampling theorem

Proposition 6

For a stopping time τ with $P(\tau \leq K) = 1$ for some $K \in \mathbb{N}$ and a martingale $(S_n)_{n\in\mathbb{N}}$ it holds that

$$E(|S_\tau|) < \infty \quad \textit{and} \quad E(S_\tau) = E(S_1).$$

Proof We consider the representation $S_\tau = \sum_{k=1}^{K} 1_{\{\tau=k\}} S_k$. It follows that

$$E(S_\tau) = \sum_{k=1}^{K} E(1_{\{\tau=k\}} S_k) = \sum_{k=1}^{K} E\big(1_{\{\tau=k\}} E(S_K|\mathcal{F}_k)\big)$$

$$= E\big(S_K \cdot \sum_{k=1}^{K} 1_{\{\tau=k\}}\big) = E(S_K) = E(S_1)$$

Optional sampling: extensions

- The converse is true as well: if $E(S_\tau) = E(S_1)$ for all bounded stopping times τ then S is a martingale.
- More generally for any two stopping times σ, τ with $P(\sigma \leq \tau \leq K) = 1$ for some $K \in \mathbb{N}$ and a martingale S one has

$$E(S_\tau | \mathcal{F}_\sigma) = S_\sigma.$$

- For a supermartingale S the optional sampling theorem becomes $E(S_\tau) \leq E(S_1)$. and $E(S_\tau | \mathcal{F}_\sigma) \leq S_\sigma$.

Options

Definition 7

Consider a traded underlying S.

- A European Call Option is the right to buy the underlying S at a future time T at price K. K is called exercise price or strike price.
- A European Put Option is the right to sell the underlying at price K
- An American Call (respectively Put) Option is the right to buy (respectively sell) the security at any future point in time $t \leq T$ at price K.

Cashflow at maturity.

- European call. The option willbe exercised only if $S(T) > K$. In this case, contract generates a profit $S(T) - K$. The total cashflow is therefore

$$C_T = \max\{S_T - K, 0\} =: (S_T - K)^+ \tag{3}$$

- Similarly, the cashflow of a put option in T is given by

$$P_T = \max\{K - S_T, 0\} =: (K - S_T)^+ .$$

Limits on the Option Value – Non-Dividend-Paying Stock

In the sequel we denote by $B(t, T)$ the value in $t \leq T$ of a zero coupon bond with nominal 1 and maturity T.

Proposition 8

Assume that the share S does not pay dividends. In an arbitrage-free market, we have the following bounds on the price C_t of a European call at time $t \leq T$ with strike K

1. $(S_t - KB(t, T))^+ \leq C_t \leq S_t$.
2. *Put-call-parity. The price at time t of a call, C_t, and a put, P_t, on the underlying S with maturity $T > t$ and strike K satisfy* $C_t = S_t - KB(t, T) + P_t$.
3. *Merton's theorem It is never optimal to exercise an American call prematurely. In particular, $C_t^A = C_t$.*

The financial market model

We work on some $(\Omega, \mathcal{F}, P)$ with $|\Omega| = K < \infty$ and $P(\{\omega_k\}) > 0$ for all $1 \leq k \leq K$.

- There is a finite number of trading dates $t = 0, 1, \ldots, N$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\ldots,N}$ describes the information available to investors. For simplicity $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_N = \mathcal{F}$.
- There are $d+1$ traded assets; $S_{t,i} = S_{t,i}(\omega) = S_i(t, \omega)$ denotes the price of asset $i \in \{0, 1, \ldots, d\}$ in t. We write

$$\boldsymbol{S} = (\boldsymbol{S}_t)_{t=0,\ldots,N} = (S_{t,0}, S_{t,1}, \ldots S_{t,d})'_{t=0,\ldots,N}\,.$$

  We assume that $\boldsymbol{S}$ is adapted to $\mathbb{F}$.
- S_0 is used as numéraire; we assume that $S_{t,0}(\omega) > 0$ $\forall\ t \in \{0, 1, \ldots, N\},\ \omega \in \Omega$ and $S_0(0) = 1$.

Summarizing, this model of a market is denoted $\mathcal{M} := \{(\Omega, \mathcal{F}, P), \mathbb{F}, \boldsymbol{S}\}$.

Trading strategies

Definition 9

An admissible trading strategy is a $(d+1)$-dimensional predictable stochastic process $\boldsymbol{\theta} = (\boldsymbol{\theta}_t) = (\theta_{t,0}, \theta_{t,1}, \ldots, \theta_{t,d})'$.

- $\theta_{t,i}$ gives the amount of security i an investor buys at time $t-1$ and which she holds over $(t-1, t]$.
- Predictability of $\boldsymbol{\theta}$ implies that choice of $\boldsymbol{\theta}_t$ at time $t-1$ is based only on $\mathcal{F}_{t-1}$. (no insider information).
- The value of the strategy $\boldsymbol{\theta}$ in t is given by

$$V_t^{\boldsymbol{\theta}} := \boldsymbol{\theta}_t' \boldsymbol{S}_t = \sum\nolimits_{i=0}^{d} \theta_{t,i} S_{t_i}.$$

- An admissible strategy $\boldsymbol{\theta}$ is called selffinancing, if for $t = 1, \ldots, N-1$ one has

$$V_t^{\boldsymbol{\theta}} = \boldsymbol{\theta}_t' \boldsymbol{S}_t \overset{!}{=} \boldsymbol{\theta}_{t+1}' \boldsymbol{S}_t. \tag{4}$$

(no intermediate injection or withdrawal of funds)

Characterization of selffinancing strategies

The so-called gains from trade (can be losses) of a selffinancing strategy $\boldsymbol{\theta}$ are given by

$$G_t^{\boldsymbol{\theta}} := (\boldsymbol{\theta}, \boldsymbol{S})_t := \sum_{u=1}^{t} \boldsymbol{\theta}_u' \Delta \boldsymbol{S}_u = \sum_{u=1}^{t} \boldsymbol{\theta}_u' (\boldsymbol{S}_u - \boldsymbol{S}_{u-1})$$

Note that $(\boldsymbol{\theta}, \boldsymbol{S})_t$ can be viewed as (discrete) stochastic integral $\int_0^t \boldsymbol{\theta}_u dS_u$.

Lemma 10 (Characterization of selffinancing strategies)

An admissible trading strategy $\boldsymbol{\theta}$ is selffinancing if and only if it holds for all $t = 1, \dots, N$ that

$$V_t^{\boldsymbol{\theta}} = V_0^{\boldsymbol{\theta}} + (\boldsymbol{\theta}, \boldsymbol{S})_t. \tag{5}$$

Proof

By definition of $V^{\boldsymbol{\theta}}$

$$V^{\boldsymbol{\theta}}_{t+1} - V^{\boldsymbol{\theta}}_t = \boldsymbol{\theta}'_{t+1}\boldsymbol{S}_{t+1} - \boldsymbol{\theta}'_t\boldsymbol{S}_t. \tag{6}$$

Now $\boldsymbol{\theta}$ is selffinancing $\Leftrightarrow$ for all t it holds that $\boldsymbol{\theta}'_t\boldsymbol{S}_t = \boldsymbol{\theta}'_{t+1}\boldsymbol{S}_t$. This gives

$$V^{\boldsymbol{\theta}}_{t+1} - V^{\boldsymbol{\theta}}_t = \boldsymbol{\theta}'_{t+1}\Delta\boldsymbol{S}_{t+1} \tag{7}$$

and hence (4)

Conversely (5) implies (7) and, using (6), we get that $\boldsymbol{\theta}$ is selffinancing. ■

Discounted Quantities

The analysis of the model simplifies if we work with discounted quantities. Recall that we assumed $S_0 > 0$. Put

$$\tilde{S}_t := \left(1, \frac{S_{t,1}}{S_{t,0}}, \dots, \frac{S_{t,d}}{S_{t,0}}\right)'$$
$$\tilde{V}_t^{\boldsymbol{\theta}} := \frac{V_t^{\boldsymbol{\theta}}}{S_{t,0}} = \boldsymbol{\theta}_t' \tilde{\boldsymbol{S}}_t$$
$$\tilde{G}_t^{\boldsymbol{\theta}} := (\boldsymbol{\theta}, \tilde{\boldsymbol{S}})_t.$$

Repeating the previous argument it is easily seen that a strategy $\boldsymbol{\theta}$ ist selffinancing iff

$$\tilde{V}_t^{\boldsymbol{\theta}} = \tilde{V}_0^{\boldsymbol{\theta}} + \tilde{G}_t^{\boldsymbol{\theta}}.$$

Alternative description of selffinancing strategies

Since $\tilde{S}_{t,0} \equiv 1$ so that $\tilde{G}^{\boldsymbol{\theta}}$ is independent of $(\theta_{t,0})_{t=1,\ldots,N}$. Hence for a selffinancing strategy initial value V_0 and the position $(\theta_{t,1},\ldots,\theta_{t,d})'_{t=1,\ldots N}$ in the "risky assets" determine uniquely the position in the numeraire asset.

Conversely,for given V_0 every strategy $(\theta_{t,1},\ldots,\theta_{t,d})$ in the "risky assets" can be made selffinancing by a proper choice of θ_0: one has

$$\theta_{t,0} = \tilde{V}_0 + \tilde{G}_t^{\boldsymbol{\theta}} - \sum_{i=1}^{d} \theta_{t,i}\tilde{S}_{t,i};$$

it is easily seen that $\theta_{t,0}$ is predictable.

Absence of Arbitrage

Definition 11

A selffinancing strategy $\boldsymbol{\theta}$ is an arbitrage opportunity, if $V_0^{\boldsymbol{\theta}} \leq 0$, $V_N^{\boldsymbol{\theta}} \geq 0$ and if at least one of the following holds

$$i) V_0^{\boldsymbol{\theta}} < 0 \quad \text{or} \quad ii) P\left(V_N^{\boldsymbol{\theta}} > 0\right) > 0.$$

A market (model) $\mathcal{M}$ is free of arbitrage (NA) if there are no arbitrage opportunities.

Two reasons for NA:

- On real markets arbitrage opportunities persist only for short time
- A pricing model that is not arbitrage free leads to inconsistent prices for derivatives and hence to losses for the user of the model

Equivalent martingale measure

Definition 12

Given a model $\mathcal{M}$ for a discrete-time financial market. A probability measure Q on $(\Omega, \mathcal{F})$ is called equivalent martingale measure if

1. $Q \sim P$, that is for all $A \in \mathcal{F}$ one has $Q(A) = 0 \;\Leftrightarrow P(A) = 0$.
2. The discounted price process $\tilde{\boldsymbol{S}}$ of all traded assets is a Q-martingale that is for all $t = 0, 1, \ldots N-1$ and all $i = 0, 1, \ldots d$.,

$$E^Q\left(\tilde{S}_{t+1,i} \,\middle|\, \mathcal{F}_t\right) = \tilde{S}_{t,i}\,. \tag{8}$$

An equivalent martingale measure is often called risk-neutral measure.

Martingale measure and selffinancing strategies

Lemma. Suppose that Q is a risk-neutral measure for the market $\mathcal{M}$. Then for every admissible selffinancing strategy $\boldsymbol{\theta}$ the corresponding discounted value process $\tilde{V}^{\boldsymbol{\theta}}$ is a martingale.

Proof. Since $\boldsymbol{\theta}$ is selffinancing we get $\tilde{V}^{\boldsymbol{\theta}}_{t+1} = \tilde{V}^{\boldsymbol{\theta}}_t + \boldsymbol{\theta}'_{t+1}\Delta\tilde{\boldsymbol{S}}_{t+1}$. Now $\boldsymbol{\theta}_{t+1}$ is $\mathcal{F}_t$-measurable as $\boldsymbol{\theta}$ is predictable. This implies that

$$\begin{aligned} E^Q\big(\tilde{V}^{\boldsymbol{\theta}}_{t+1} \mid \mathcal{F}_t\big) &= \tilde{V}^{\boldsymbol{\theta}}_t + \sum_{i=1}^{d} E^Q\Big(\theta_{t+1,i}(\tilde{S}_{t+1,i} - \tilde{S}_{t,i}) \,\Big|\, \mathcal{F}_t\Big) \\ &= \tilde{V}^{\boldsymbol{\theta}}_t + \sum_{i=1}^{d} \theta_{t+1,i} \underbrace{E^Q\big(\tilde{S}_{t+1,i} - \tilde{S}_{t,i} \mid \mathcal{F}_t\big)}_{=0 \text{ as } \tilde{\boldsymbol{S}} \text{ is a } Q \text{ martingale}} = \tilde{V}^{\boldsymbol{\theta}}_t . \end{aligned}$$

∎

1st Fundamental Theorem of Asset Pricing

Proposition 13

If an equivalent martingale-measure exists for the security market model $\mathcal{M}$, the model $\mathcal{M}$ is arbitrage-free.

Proof. Consider $\boldsymbol{\theta}$ selffinancing with $V_N^{\boldsymbol{\theta}} \geq 0,\ P(V_N^{\boldsymbol{\theta}} > 0) > 0$; we want to show that existence of Q implies that $V_0^{\boldsymbol{\theta}} > 0$.

Now $\operatorname{sign}(V_N^{\boldsymbol{\theta}}) = \operatorname{sign}(\tilde{V}_N^{\boldsymbol{\theta}})$, $\Rightarrow$ we have $\tilde{V}_N^{\boldsymbol{\theta}} \geq 0$, $P(\tilde{V}_N^{\boldsymbol{\theta}} > 0) > 0$.

As $Q \sim P$ we also have $Q(\tilde{V}_N^{\boldsymbol{\theta}} > 0) > 0$ and hence $E^Q(\tilde{V}_N^{\boldsymbol{\theta}}) > 0$. Recall that $(\tilde{V}_n^{\boldsymbol{\theta}})_{n=0,\ldots,N}$ is a Q-martingale. Hence

$$V_0^{\boldsymbol{\theta}} = \tilde{V}_0^{\boldsymbol{\theta}} = E^Q(\tilde{V}_N^{\boldsymbol{\theta}}) > 0.$$

$\blacksquare$

Alternative characterization of the martingale property

Now we want to establish the converse direction. For this we need the following characterization of the martingale property of $\tilde{\boldsymbol{S}}$.

Lemma The discounted price process $\tilde{S}$ is a Q-martingale if and only if it holds for any selffinancing trading strategy that

$$E^Q\left(\tilde{G}_N^{\boldsymbol{\theta}}\right) = 0.$$

Proof. $\Rightarrow$: Obviously, $\tilde{G}^{\boldsymbol{\theta}}$ is the discounted value process of the strategy with initial investment $V_0 = 0$ and position $\theta_{t,1}, \ldots \theta_{t,d}$ in the "risky assets". If $\tilde{S}$ is a Q-martingale, $\tilde{G}^{\boldsymbol{\theta}}$ is a Q-martingale and it follows $E^Q\left(\tilde{G}_N^{\boldsymbol{\theta}}\right) = \tilde{G}_0^{\boldsymbol{\theta}} = 0$.

Converse direction.

Idea. By assumption $E^Q(\tilde{G}_N^{\boldsymbol{\theta}}) = 0$ for all $\boldsymbol{\theta}$. A clever choice of $\boldsymbol{\theta}$ then gives the martingale property of $\tilde{\boldsymbol{S}}$ (in the following $d = 1$ for simplicity):

Recall that $\tilde{S}$ is a martingale if (and only if) $\tilde{S}_t \in L^1$ and $E^Q(\tilde{S}_{t+1} 1_A) = E^Q(\tilde{S}_t 1_A)$ for all $A \in \mathcal{F}_t$ and $t = 0, \ldots, N-1$. Now consider a selffinancing strategy with $V_0 = 0$ and

$$\theta_{s,1}(\omega) = 1_A(\omega) 1_{\{s=t+1\}}$$

(buy one unit at t if A occurs and sell in $t+1$). It follows that

$$G_N^{\theta} = (\tilde{S}_{t+1} - \tilde{S}_t) 1_A.$$

By assumption we have

$$0 = E^Q\big(G_N^{\theta}\big) = E^Q\big((\tilde{S}_{t+1} - \tilde{S}_t) 1_A\big). \qquad \blacksquare$$

1st fundamental theorem of asset pricing

Theorem 14

The security market model $\mathcal{M}$ is arbitrage-free if and only if there exists at least one equivalent martingale measure.

Comment.

- Famous result that holds (with additional technicalities) in general continuous-time models. Various results due to Kreps, [Dalang et al., 1990], [Delbaen and Schachermayer, 1994]. A good overview can be found in Chapter 10 of [Björk, 2004].

Risk neutral pricing

Now we turn to the pricing of contingent claims. Formally, a contingent claim with maturity $T \in \{1, \ldots, N\}$ is simply an $\mathcal{F}_T$-measurable rv (eg a derivative).

Definition. Consider a model $\mathcal{M}$ for a financial market.

- A contingent claim H with maturity T is called attainable if there is a selffinancing strategy $\boldsymbol{\theta}$ with $V_T^{\boldsymbol{\theta}} = H$.; $\boldsymbol{\theta}$ is called replicating strategy for H.
- Suppose that $\mathcal{M}$ is arbitrage-free and H attainable with replicating strategy $\boldsymbol{\theta}$. Then the fair price of H in $t \leq T$ is given by the cost $V_t^{\boldsymbol{\theta}}$ of the replicating portfolio.
- The market $\mathcal{M}$ is called complete, if any contingent claim H is attainable.

Comment. A replicating strategy can be used to mitigate (in theory even to eliminate) the risk incurred by selling the claim H.

Risk-neutral pricing

Theorem. Consider a contingent claim H with candidate price process $\Pi(t; H)$, $t = 0, 1, \ldots, N$.

1. If the extended model with price processes $\big(S^0, S^1, \ldots S^d, \Pi(\cdot, H)\big)$ is arbitrage free we must have

$$\Pi(t, H) = S_t^0 E^Q\left(\frac{H}{S_T^0}|\mathcal{F}_t\right) \tag{9}$$

   for some martingale measure Q for the market $(S^0, \ldots, S^d)$.

2. If H is attainable, we have for any martingale measure Q the relation $S_t^0 E^Q\left(\frac{H}{S_T^0}|\mathcal{F}_t\right) = V_t^{\boldsymbol{\theta}}$, where $\boldsymbol{\theta}$ is a replicating strategy for H.

Comments.

- If S_0 is a bank account with one-period interest rate (r_t), so

$$\Pi(t, H) = E^Q\Big(\prod_{s=t+1}^{T} \frac{1}{1+r_s} \cdot H \mid \mathcal{F}_t \Big).$$

- Condition (9) is necessary to ensure that there is some martingale measure for the extended market.
- Two different martingale measure Q_1, Q_2 generally lead to two different price processes $\Pi_1(t, H)$, $\Pi_2(t, H)$.
- The fair arbitrage-free price of an attainable claim is uniquely defined.
- If the market is complete, we must have for any two martingale measures Q_1 and Q_2 and any bounded random variable H that $E^{Q_1}(H) = V^{\theta}(0) = E^{Q_2}(H)$

Second fundamental theorem of asset pricing

Theorem 15 (Second fundamental theorem of asset pricing)

Consider a market that admits (at least) one martingale measure Q. Then the market is complete if and only if Q is unique.

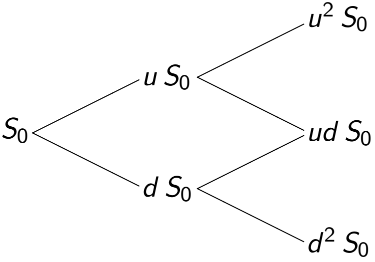
Proof. $\Rightarrow$ is a straightforward implication of the risk neutral pricing rule. For the converse direction one considers a non-attainable claim and constructs a second risk-neutral measure.

The Binomial Model

- Binomial model was first proposed by [Cox et al., 1979].
- A simple model that illustrates basic principles of discrete-time finance
- Popular with practitioners as numerical tool as it can be used to approximate the continuous Black-Scholes model.

Informal description

- Two assets, stock S (risky) and bond (riskless) with price $B_t = (1+r)^t$
- Only two possibilities for stock price evolution: up $(S_{t+1} = uS_t)$ and down $((S_{t+1} = dS_t)$
- We assume $u > 1 + r > d > 0$



Formal description

- Let $\Omega = \{u, d\}^N$ and denote by $\xi_i(\omega)$ the ith component of $\omega \in \Omega$.
- Let

$$S_t(\omega) = S_0 \prod_{i=1}^{t} \xi_i \quad t = 0, \ldots, N. \tag{10}$$

- The filtration $\mathbb{F}$ is given by

$$\mathcal{F}_t := \sigma\left(S_1, \ldots, S_t\right) = \sigma\left(\xi_1, \ldots, \xi_t\right). \tag{11}$$

- The historical probability measure P is arbitrary, we only require $P(\{\omega\}) > 0 \ \forall \omega \in \Omega$.

The equivalent martingale measure

Consider first the one-period case. Set $q := Q(\omega_1 = u)$ Then we must have

$$S_0 = E^Q\left(\frac{1}{1+r}S_1\right) = \frac{1}{1+r}\left(qS_0u + (1-q)S_0d\right),$$

leading to

$$q = \frac{1+r-d}{u-d} \tag{12}$$

Note that $q \in (0,1)$ iff $d < 1+r < u$, which holds by assumption. Note further that q is uniquely determined.

Multi-period case

Q can be determined recursively: Let $Q(\omega_{t+1} = u|\mathcal{F}_t) := q_t$ and $Q(\omega_{t+1} = d|\mathcal{F}_t) = 1 - q_t$. Then the requirement

$$S_t = E^Q\left(\frac{1}{1+r}S_{t+1} \mid \mathcal{F}_t\right) = \frac{1}{1+r}\left(q_t S_t u + (1-q_t)S_t d\right)$$

leads to $q_t = q = \frac{1+r-d}{u-d}$. Letting

$$j_t(\omega) := \#\{i <= t :\ \omega_i = u\}. \tag{13}$$

we thus get the following expression for the equivalent martingale measure Q:

$$Q(\{\omega\}) = q^{j_t(\omega)}(1-q)^{t-j_t(\omega)}. \tag{14}$$

The above argument shows that Q is unique so that the model is complete. Note further that up-and down movements in different periods are independent under Q.

Replicating strategies

By the second fundamental theorem every contingent claim is attainable. We now compute the replicating strategy via backward induction. for the case of a claim H with maturity $T = 2$.

$t = 1.$ Assume that we are in the up-state. Then the replicating strategy $\boldsymbol{\theta}_2(u) = (\theta_{2,0}(u), \theta_{2,1}(u))'$ must satisfy the relations

$$\begin{aligned} \theta_{2,0}(u) \cdot (1+r)^2 + \theta_{2,1}(u) \cdot u\, S_1(u) &= H(u,u) \\ \theta_{2,0}(u) \cdot (1+r)^2 + \theta_{2,1}(u) \cdot d\, S_1(u) &= H(u,d). \end{aligned}$$

Solving this system of equations gives

$$\theta_{2,1}(u) = \frac{H(u,u) - H(u,d)}{S_1(u)\,(u-d)} = \frac{H(u,u) - H(u,d)}{S_0\, u(u-d)} \tag{15}$$

$$\theta_{2,0}(u) = \frac{u\, H(u,d) - d\, H(u,u)}{(u-d)(1+r)^2}. \tag{16}$$

Replicating strategies ctd.

By the risk-neutral pricing rule the value of the replicating strategy "in the up-state" at $t = 1$ equals

$$V_1(u) = \frac{1}{1+r}\left(q\, H(u,u) + (1-q)\, H(u,d)\right).$$

The case $\omega_1 = d$ is treated similarly.

$t = 0$. Here we apply the above argument to the claim $V_1(\omega_1)$. This leads to the following equations for $\boldsymbol{\theta}_0$.

$$\begin{aligned} \theta_{1,0}(1+r) + \theta_{1,1}\, u\, S_0 &= V_1(u) \\ \theta_{1,0}(1+r) + \theta_{1,1}\, d\, S_0 &= V_1(d), \end{aligned}$$

This gives

$$\theta_{1,1} = \frac{V_1(u) - V_1(d)}{S_0(u-d)} \quad \text{and} \quad \theta_{1,0} = \frac{u\, V_1(d) - d\, V_1(u)}{(1+r)(u-d)}.$$

Note that $\theta_{t,1}$ can be viewed as discrete derivative of the value wrt the stock price.

Options in the CRR model

We may compute prices of specific options using the risk-neutral pricing formula. For instance we get for a European call

Proposition. In a binomial CRR model with up-state-return u, down-state return d and interest-rate r such that $u > 1 + r > d$ the arbitrage price C_n at $t = n$ of a European call with strike price K and maturity N equals

$$C_n = \frac{1}{(1+r)^{N-n}} \sum_{j=0}^{N-n} \binom{N-n}{j} q^j (1-q)^{N-n-j} \left(S_n(\omega) u^j d^{N-n-j} - K \right)^+ .$$

Optimal Stopping Problems

The problem. Consider some filtered probability space $(\Omega, \mathcal{F}, P)$, $\mathbb{F} = (\mathcal{F}_n)_{n=0,1,\dots,N}$ and a nonnegative, adapted process $H = (H_n)_{n=0,1,\dots,N}$ (the payoff process) and let

$$\mathbb{T} := \Big\{\tau : \Omega \to \{0, 1, \dots, N\},\ \tau \text{ is an } \mathbb{F} \text{ stopping time}\Big\} \tag{17}$$

The associated optimal stopping problem is to find $\tau^* \in \mathbb{T}$ with

$$E(H_{\tau^*}) = \sup\left\{E(H_\tau) :\ \tau \in \mathbb{T}\right\} =: U^*. \tag{18}$$

Comments

- $H_n(\omega)$ gives the payoff if the system is 'stopped" at $t = n$ (eg. discounted payoff of an American option or discounted value of an investment opportunity)
- U^* is called value of the stopping problem.

Examples

- American put option. Here $H_n = B_n^{-1}(K - S_n)^+$.
- European put option with maturity $T \leq n$ Here $H_n = B_n^{-1}(K - S_n)^+ 1_{\{n=T\}}$.
- The case where H is a martingale. By the optional sampling theorem $E(H_\tau) = H_0$ for any $\tau \in \mathbb{T}$, so that any $\tau \in \mathbb{T}$ is optimal and $U^* = H_0$.

The Snell envelope

The Snell envelope is an important tool in the analysis of optimal stopping problems.

Definition. The Snell envelope $(U_n)_{n=0,1,\ldots,N}$ of the payoff process H is defined recursively by $U_N := H_N$ and

$$U_n := \max\{H_n,\ E(U_{n+1}|\mathcal{F}_n)\} \text{ for } n = N-1, N-2, \ldots, 0. \tag{19}$$

Proposition. The Snell envelope U for the payoff process H is the smallest supermartingale $\tilde{U}$ such that $\tilde{U}_n \geq H_n$ for all $n \in \{0, 1, \ldots N\}$.

Characterization of optimal stopping times

Theorem. Consider an optimal stopping problem with payoff process H and associated Snell envelope U. Then the following holds.

1. A stopping time $\tau \in \mathbb{T}$ is optimal if and only if the following two conditions hold
   (i) $H_\tau = U_\tau$
   (ii) The stopped process U^τ with $U^\tau_n := U_{\tau \wedge n}$ is a martingale.
2. An optimal stopping time is given by
   $\tau^* = \tau_{min} := \inf\{n = 0, \ldots, N : \; U_n = H_n\}$.
3. The Snell envelope gives the value of the problem i.e. $U^* = U_0$.

Proof: key steps

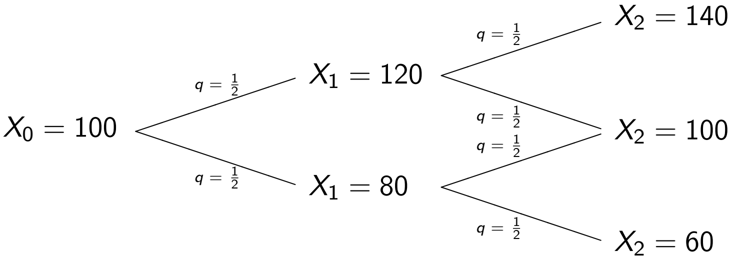
a) Since U is a supermartingale with $U_\tau \geq H_\tau$ we have the inequalities

$$E(H_{\tilde{\tau}}) \leq E(U_{\tilde{\tau}}) = E(U_N^{\tilde{\tau}}) \leq U_0, \quad \forall \tilde{\tau} \in \mathbb{T}, \tag{20}$$

b) If τ satisfies (i) and (ii) we have equality in (20) and τ is optimal. c) Using the definition of the Snell envelope it is easily seen that τ_{min} satisfies (i) and (ii), so that τ_{min} is optimal. It follows that equality holds in (20) for $\tilde{\tau} = \tau_{min}$ which gives $U^* = U_0$.

Example (stylized call spread)

Consider a state variable X in a two-period model that evolves according to the following tree



We consider a call spread with $K_1 = 95$ and $K_2 = 120$; this gives the payoff process $H_n = (X_n - 95)^+ - (X_n - 120)^+$.

Example (stylized call spread)ctd.

The following table gives the values of H and U for every path ω.

path		$n=0$			$n=1$			$n=2$
ω_1	$X_0=100$	$H_0=5$	$U_0=13.75$	$X_1=120$	$H_1=25$	$U_1=25$	$X_2=140$	$H_2=U_2=25$
ω_2	$X_0=100$	$H_0=5$	$U_0=13.75$	$X_1=120$	$H_1=25$	$U_1=25$	$X_2=100$	$H_2=U_2=5$
ω_3	$X_0=100$	$H_0=5$	$U_0=13.75$	$X_1=80$	$H_1=0$	$U_1=2.5$	$X_2=100$	$H_2=U_2=5$
ω_4	$X_0=100$	$H_0=5$	$U_0=13.75$	$X_1=80$	$H_1=0$	$U_1=2.5$	$X_2=60$	$H_2=U_2=0$

It follows that the optimal stopping time is

$$\tau_{min}(\omega)=\inf\Big\{n\in\{0,1,2\}:\ U_n(\omega)=H_n(\omega)\Big\}=\begin{cases}1 & \text{for } \omega_1=u\\ 2 & \text{for } \omega_1=d\,.\end{cases}$$

Application to American options

Here one can show the following:

Theorem 16

The fair price of an American claim with payoff process $(C_t)_{t=0,1,\dots,N}$ in an arbitrage-free and complete security market $\mathcal{M}$ with martingale measure Q is given by

$$V_0^{\theta} = S_{0,0} U_0^{\tilde{C}},$$

where $(U_t^{\tilde{C}})_{t=0,1,\dots,N}$ is the Snell envelope of the discounted payoff process $(\tilde{C}_t)_{t=0,1,\dots,N} = (\frac{C_t}{S_{t,0}})_{t=0,1,\dots,N}$.

Overview

Why continuous time models

Main theme of the lecture: extension of derivative pricing to continuous time. This involves a substantial mathematical effort, but there are good reasons for this:

- reference case for discrete high frequency trading
- explicit formulas instead of approximations (eg. Black Scholes formula)
- standard in the (math) finance literature and in industry

But discrete-time models remain important as numerical tools.

Stochastic Processes in Continuous Time

Consider a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$. A continuous-time stochastic process $X = (X_t)_{t \geq 0}$ is a family of random variables on $(\Omega, \mathcal{F}, P)$.

- X is called adapted, if for all $t > 0$ the rv X_t is $\mathcal{F}_t$-measurable.
- The marginal distribution of the process at a given $t \geq 0$ is the distribution $\mu(t)$ of the rv X_t.
- Consider time points $(t_1, ..., t_n)$ in $[0, \infty)$. Then $(X_{t_1}, ..., X_{t_n})$ is a random vector with values in $\mathbb{R}^n$ and distribution $\mu(t_1, ..., t_n)$, The class of all such distributions is the set of finite-dimensional distributions of X.
- Fix some $\omega \in \Omega$. The mapping

$$X_{\cdot}(\omega)\colon [0, \infty) \to \mathbb{R}, \quad t \to X_t(\omega)$$

  is called trajectory or sample path of X. We will consider processes with continuous sample paths (eg. Brownian motion) or right continuous with left limits (RCLL) such as the Poisson process.

Classes of stochastic processes

Definition 17 (Martingales.)

An adapted stochastic process X with $E(|X_t|) < \infty$ for all $t > 0$ is

- a submartingale if $\forall t, s$ with $t > s$ we have $E(X_t|\mathcal{F}_s) \geq X_s$.
- a supermartingale if $\forall t, s$ with $t > s$ we have $E(X_t|\mathcal{F}_s) \leq X_s$.
- a martingale if X is both a sub- and a supermartingale, i.e. if $E(X_t|\mathcal{F}_s) = X_s$ for all t, s.

Semimartingales. Loosely speaking a semimartingale is an adapted process X of the form $X_t = M_t + A_t$ such that

- M is a martingale (the unpredictable noise part of X)
- A is of the form $A_t = \int_0^t a_s ds$ for some adapted process $(a_s)_{s \geq 0}$. (A is the systematic part and a_t is the local trend of the process in t.)

Markov-Processes

An adapted stochastic process X is called Markov process, if for all $t, s > 0$ and all bounded functions $f\colon \mathbb{R} \to \mathbb{R}$

$$E(f(X_{t+s}) \mid \mathcal{F}_t) = E(f(X_{t+s}) \mid \sigma(X_t)). \tag{21}$$

Intuitively X is Markov if the conditional distribution of future values X_{t+s}, $s \geq 0$, is completely determined by the present value X_t (no memory); in particular past values X_u, $u < t$ of the process do not contain any additional information which is useful for predicting X_{t+s}.

Point processe and the Poisson process

Point processes. Assume that certain relevant 'events' (eg defaults of counterparties in a financial context) occur at random time points $\tau_0 < \tau_1 < \dots$. The corresponding point process N_t is then given by $N_t := \sup\{n, \tau_n \le t\}$, i.e. N_t measures the number of events which have occurred up to time t.

Poisson process. The Poisson process is a special point process. Mathematical construction: take a sequence Y_n of independent $Exp(\lambda)$-distributed random variables with $P(Y_n \ge x) = e^{-\lambda x}$. Let

$$\tau_n := \sum_{j=1}^{n} Y_j .$$

(Y_n is the waiting time between event $n-1$ and event n.) The process $N_t = \sup\{n \colon \tau_n \le t\}$ is then a Poisson process with intensity λ.

Properties of the Poisson process

The Poisson process has among others the following properties

- $P(N_t = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$, $k = 0, 1, \ldots$, $t \geq 0$.
- $N_{t+u} - N_t$ is independent of N_s for $s \leq t$ and Poisson-distributed with parameter (λu).
- The compensated Poisson process $M_t := N_t - \lambda t$ is a martingale; in particular $E(N_t) = \lambda t$.

Compound Poisson process. Consider a sequence of iid variables Z_i, $i \in \mathbb{N}$ and a Poisson process N. Then

$$S_t = \sum_{i=1}^{N_t} Z_i$$

is called a compound Poisson process. (an important building block for insurance models)

Stopping Times

Consider a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}$.

Definition 18

A rv $\tau\colon \Omega \to [0, \infty]$ is called stopping time wrt. $\{\mathcal{F}_t\}$ if for all $t \geq 0$ it holds that $\{\tau \leq t\} \in \mathcal{F}_t$.

Example: First hitting times. Given a stochastic process X and a Borel set A in $\mathbb{R}$. Define $\tau_A := \inf\{t \geq 0\colon X_t \in A\}$. Then the rv τ_A is called first hitting time into the set A. It can be shown that τ_A is a stopping time if X is adapted.

The sigma-field $\mathcal{F}_\tau$.

Definition 19

Given a stopping time τ. Then we call

$$\mathcal{F}_\tau := \left\{ A \in \mathcal{F} \colon A \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for all } t \ge 0 \right\} \tag{22}$$

the σ-field of the events which are observable up to τ.

Let $(X_t)_{t \ge 0}$ be a right-continuous stochastic process and let τ be a stopping time. We define the stopped rv X_τ by

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega) \cdot 1_{\{\tau < \infty\}}(\omega), \quad \omega \in \Omega. \tag{23}$$

Application to stochastic processes

Let $(X_t)_{t\geq 0}$ be an $\{\mathcal{F}_t\}$-adapted and right-continuous stochastic process and let τ be a stopping time. Then we have the following

- The rv X_τ is $\mathcal{F}_\tau$-measurable.
- Define the stopped process $X^\tau = (X_t^\tau)_{t\geq 0}$ by

$$X_t^\tau(\omega) := X_{t\wedge\tau(\omega)}(\omega) = \begin{cases} X_{\tau(\omega)}(\omega), & \tau(\omega) \leq t. \\ X_t(\omega), & \tau(\omega) > t. \end{cases} \tag{24}$$

  Then the stopped process X^τ is adapted.

The optional sampling theorem

Theorem 20 (Optional sampling theorem)

Consider an adapted stochastic process $X = (X_t)_{t \geq 0}$ with $E(|X_t|) < \infty$, $t \geq 0$. Then the following are equivalent.

(1) *X is a martingale.*

(2) *For all bounded stopping times τ ($\tau(\omega) \leq C$ for some $C > 0$, all $\omega \in \Omega$) one has $E(X_\tau) = E(X_0)$.*

(3) *Given two stopping times S and T such that $S \leq T \leq C$ for some $C > 0$. Then $E(X_T \mid \mathcal{F}_S) = X_S$.*

Corollary 21

Let X be a martingale with right-continuous trajectories and let τ be a stopping time. Then the stopped process X^τ with $X^\tau_t = X_{t \wedge \tau}$ is also a martingale.

Proofs can be found in [Protter, 2005]

Brownian motion

Brownian motion is a key building block for most continuous-time models in finance.

Definition 22

A stochastic process $X = (X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, P)$ is standard one-dimensional Brownian motion, if

(i) $X_0 = 0$ a.s.

(ii) X has independent increments: for all $t, u \geq 0$ the increment $X_{t+u} - X_t$ is independent of X_s for all $s \leq t$.

(iii) X has stationary, normally distributed increments: $X_{t+u} - X_t \sim N(0, u)$.

(iv) X has continuous sample paths.

In honor of the botanist Brown and and the mathematician Norbert Wiener Brownian motion is often denoted by $(B_t)_{t \geq 0}$ or by $(W_t)_{t \geq 0}$.

Elementary properties of Brownian motion

(i) $W_t = W_t - W_0$ is $N(0, t)$-distributed.

(ii) Let $t > s$. Then the covariance of W_t and W_s is given by $\mathrm{cov}(W_t, W_s) = E(W_t W_s) = E\big((W_t - W_s)W_s\big) + E(W_s^2) = E(W_t - W_s)E(W_s) + s = s$.

(iii) The finite-dimensional distributions of W are multivariate normal distributions with mean 0 and covariance matrix given in (ii).

Construction of Brownian motion

The construction of Brownian motion is important for simulation. One possibility is via Donskers invariance principle.

Theorem 23

Consider iid rvs X_i with $E(X_i) = 0$ $\mathrm{var}(X_i) = 1$. Put $t_0 = 0$, $t_1 = \frac{1}{n}$, $t_2 = \frac{2}{n}$, ... Let

$$W^n_{t_j} = \frac{1}{\sqrt{n}} \sum_{i=1}^{j} X_i,$$

and set $W^n_t = W^n_{t_j}$ for $t \in [t_j, t_{j+1})$. Then W^n converges in distribution to a Brownian motion.

Some stochastic properties of Brownian motion

Proposition 24

Let W_t be standard Brownian motion and define $\mathcal{F}_t := \sigma(W_s, s \leq t)$. Then

a) $(W_t)_{t \geq 0}$
b) $(W_t^2 - t)_{t \geq 0}$ *and*
c) $\exp(\sigma W_t - 1/2\sigma^2 t)$

are martingales with respect to the filtration $\{\mathcal{F}_t\}$.

Proof is an exercise.

First and Quadratic Variation

Fix some horizon $\overline{T}$. A partition τ of $[0, \overline{T}]$ is a set of time-points $t_0 = 0 < t_1 < \ldots < t_n = \overline{T}$. The mesh of this partition is given by $|\tau| := \sup_{1 \leq i \leq n} |t_i - t_{i-1}|$.

Definition 25 (First Variation)

Consider a function $X : [0, \overline{T}] \to \mathbb{R}$. The first variation of X on $[0, \overline{T}]$ is defined as

$$\mathrm{Var}(X) := \sup \left\{ \sum_{t_i \in \tau} |X(t_i) - X(t_{i-1})|, \ \tau \text{a partition of } [0, \overline{T}] \right\}$$

If $\mathrm{Var}(X) < \infty$ X is said to be of finite variation (FV).

Remark. Suppose that $X\colon [0, \overline{T}] \to \mathbb{R}$ is continuously differentiable. Then X is of finite variation.

Quadratic Variation

Definition 26 (Quadratic Variation)

Consider a function $X : [0, \overline{T}] \to \mathbb{R}$ and a sequence τ_n of partitions of $[0, \overline{T}]$ such that $|\tau_n| \to 0$ as $n \to \infty$. Define for $t \in [0, \overline{T}]$ the quadratic variation of X along the partition τ_n by

$$V_t^2(X; \tau_n) := \sum_{t_i \in \tau_n;\, t_i < t} (X(t_i) - X(t_{i-1}))^2 \,.$$

Assume that for all $t \in [0, \overline{T}]$ the limit $[X]t := \lim_{n \to \infty} V_t^2(X; \tau_n)$ exists In that case X is said to admit the quadratic variation $[X]_t$. If the function $t \to [X]_t$ is continuous X has a continuous quadratic variation.

Note that $[X]_t$ is increasing in t.

First and Quadratic Variation

Proposition 27

If $X : [0, \overline{T}] \to \mathbb{R}$ is continuous and of finite variation its quadratic variation $[X]_t$ is zero.

By negating this result we have

Corollary 28

If X is continuous and if the function $t \to [X]_t$ is strictly increasing, X is of infinite first variation on every subinterval $[a, b]$ of $[0, \overline{T}]$.

First and Quadratic Variation ctd

Proof of the proposition.

Choose a sequence of partitions τ_n of $[0, \overline{T}]$ such that $\lim_{n\to\infty} |\tau_n| = 0$. Then

$$\begin{aligned}\sum_{t_i\in\tau_n;\, t_i\le t} (X(t_i) - X(t_{i-1}))^2 &\le \sup_{t_i\in\tau_n} |X(t_i) - X(t_{i-1})| \sum_{t_i\in\tau_n} |X(t_i) - X(t_{i-1})| \\ &\le \sup_{t_i\in\tau_n} |X(t_i) - X(t_{i-1})| \operatorname{Var}(X). \qquad (25)\end{aligned}$$

Now note that $\operatorname{Var}(X) < \infty$ and that $\sup_{t_i\in\tau_n} |X(t_i) - X(t_{i-1})| \to 0$ for $n \to \infty$ as X is continuous and as $\lim_{n\to\infty} |\tau_n| = 0$. Hence the right side of (25) converges to zero.

Quadratic Variation of a semimartingale

The following result shows that the quadratic variation of the sample paths of a continuous semimartingale is determined by the quadratic variation of its martingale part.

Proposition 29

Assume that X is continuous with quadratic variation $[X]_t$ and consider a continuous function $A : [0, \overline{T}] \to \mathbb{R}$ which is of finite first variation. Let $Y = X + A$, that is $Y_t = X_t + A_t$ for all t. Then we have $[Y]_t = [X]_t$.

Quadratic Variation of Brownian motion

Theorem 30

Consider a sequence of partitions τ_n *of* $[0, \overline{T}]$ *such that* $\lim_{n\to\infty} |\tau_n| = 0$*. Then we have for all* $t \in [0, \overline{T}]$ *that* $E\left(V_t^2(B_\cdot(\omega); \tau_n) - t\right)^2 \to 0$ *as* $n \to \infty$*.*

Implications

- $V_t^2(B_\cdot(\omega); \tau_n)$ converges to t in probability.
- Sample paths of Brownian motion are of infinite first variation (and in fact nowhere differentiable).

Overview

Introduction

In a discrete time model with trading dates $t_1, \dots, t_N$ the gains from trade of some trading strategy θ are given by

$$G^\theta_{t_N} = \sum_{i=1}^{N} \theta_{t_i}(S_{t_i} - S_{t_{i-1}})$$

In continuous-time it is natural to define the gains from trade of a trading strategy $(\theta_t)_{t \le T}$ as limit of the above expression for $N \to \infty$ and $t_n - t_{n-1} \to 0$. This is not straightforward:

Theorem. Suppose that $\lim_{N\to\infty} G^\theta_{t_N}$ exists for all continuous functions $\theta : [0, T] \to \mathbb{R}$. Then S has trajectories of finite variation.

This problem is addressed in the Ito calculus by two approaches:

- reduce the space of integrands
- use martingale theory

The Itô formula

Theorem 31 (Itô's formula)

Consider a continuous function $X : [0, T] \to \mathbb{R}$ *with continuous quadratic variation* $[X]_t$ *and a twice continuously differentiable function* $F : \mathbb{R} \to \mathbb{R}$*. Then we have for* $t \leq \overline{T}$

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)dX_s + \frac{1}{2}\int_0^t F''(X_s)d[X]_s \tag{26}$$

where

$$\int_0^t F'(X_s)dX_s := \lim_{n\to\infty} \sum_{t_i \in \tau_n;\, t_i \leq t} F'(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}). \tag{27}$$

The existence of the limit in (27) is shown in the proof of the theorem. The integral $\int_0^t F'(X_s)dX_s$ is called Itô-integral.

Comments on Itô's formula

- If X is of finite variation (and hence $[X]_t \equiv 0$) the Ito formula reduces to the classical change of variable formula

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s$$

  which holds for all f that are $\mathcal{C}^1$. The correction term $\frac{1}{2}\int_0^t F''(X_s)d[X]_s$ is crucial for many results in (mathematical) finance.
- The sums used in defining the Itô-integral are non-anticipating, i.e. the integrand $F'(X_s)$ is evaluated at the left boundary of the interval $[t_{i-1}, t_i]$;
- Often the Ito formula is expressed in following shorthand-notation: $dF(X_t) = F'(X_t)dX_t + \frac{1}{2}F''(X_t)d[X]_t$.

Some Examples

1) Take $F(x) = x^n$. Applying the Itô-formula yields

$$X_t^n = X_0^n + n \int_0^t X_s^{n-1} dX_s + \frac{n(n-1)}{2} \int_0^t X_s^{n-2} d[X]_s \,.$$

In short notation: $dX_t^n = nX_t^{n-1} dX_t + \frac{n(n-1)}{2} X_t^{n-2} d[X]_t$. If X is a sample path of a Brownian motion B with $B_0 = 0$ we obtain

$$B_t^2 = 2 \int_0^t B_s dB_s + \int_0^t d[B]_s = 2 \int_0^t B_s dB_s + t$$

2) Take $F(x) = e^x$. We get $e^{X_t} = e^{X_0} + \int_0^t e^{X_s} dX_s + \frac{1}{2} \int_0^t e^{X_s} d[X]_s$, or in short notation $de^{X_t} = e^{X_t} dX_t + \frac{1}{2} e^{X_t} d[X]_t$.

A tool for the proof

Lemma For a piecewise continuous function $g : [0, T] \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \sum_{t_i \in \tau_n;\, t_i \le t} g(t_{i-1})(X_{t_i} - X_{t_{i-1}})^2 = \int_0^t g(s)d[X]_s. \tag{28}$$

Proof of the Lemma. Recall the definition of $V_t^2(X;\tau_n)$ in Definition 26. For indicator functions of the form $g(t) = 1_{(a,b]}(t)$ the convergence in (28) translates as

$$\lim_{n\to\infty} \left(V_b^2(X;\tau_n) - V_a^2(X;\tau_n)\right) = [X]_b - [X]_a\,,$$

which is satisfied by definition. For a general function g the claim follows if we approximate g by piecewise constant functions.

Quadratic Variation of the Ito integral

Lemma 32

Consider a continuous function $X(t)$ with continuous quadratic variation $[X]_t$ and $F \in \mathcal{C}^1(\mathbb{R})$. Then the function $t \to F(X_t)$ has quadratic variation $\int_0^t (F'(X_s))^2 \, d[X]_s$.

Applying this to $F = \int f(x)dx$ we get

Theorem 33

For $f \in \mathcal{C}^1(\mathbb{R})$ the Itô-integral $I_t := \int_0^t f(X_s)dX_s$ is well-defined; its quadratic variation equals $[I]_t = \int_0^t f^2(X_s)d[X]_s$.

Example. We compute $[B^2]_t$. Itos formula gives $B_t^2 = \int_0^t 2B_s dB_s + t$. Define $I_t := \int_0^t 2B_s dB_s$. We get $[B^2]_t = [I]_t = \int_0^t 4B_s^2 ds$.

Martingale property of the Ito integral

If M is a martingale with continuous trajectories of continuous quadratic variation and f a $\mathcal{C}^1$ function we expect that $I_t := \int_0^t f(M_s)dM_s$ to inherits the martingale property from M, as

$$I_t = \lim_{n\to\infty} I_t^n \text{ with } I_t^n = \sum_{t_i\in\tau_n;\, t_i\le t} f(M_{t_{i-1}})(M_{t_i} - M_{t_{i-1}})\,.$$

This is nearly true; we will see that I_t is a local martingale.

Definition 34

A stochastic process M is called a local martingale, if there are stopping times $T_1 \le \ldots \le T_n \le \ldots$ such that

(i) $\lim_{n\to\infty} T_n(\omega) = \infty$ a.s.

(ii) $(M_{T_n\wedge t})_{t\ge 0}$ is a martingale for all n.

Martingale property of the Ito integral ctd

Theorem 35

Consider a local martingale M with continuous trajectories and continuous quadratic variation $[M]_t$ and a function $f \in \mathcal{C}^1(\mathbb{R})$. Then $I_t(\omega) = \int_0^t f(M_s(\omega))dM_s(\omega)$ is a local martingale.

Partial proof. Let τ_n be a sequence of partitions with $|\tau_n| \to 0$, and fix n. Then the discrete-time process $I_k^n := \sum_{t_i \in \tau_n, i \leq k} f(M_{t_{i-1}})(M_{t_i} - M_{t_{i-1}})$, $k \leq n$, is a martingale wrt the discrete filtration $\{\mathcal{F}_k^n\}_k$ with $\mathcal{F}_k^n := \mathcal{F}_{t_k}$, as can be seen from the following easy argument.

$$\begin{aligned} E(I_k^n - I_{k-1}^n | \mathcal{F}_{k-1}^n) =& E(f(M_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}) | \mathcal{F}_{t_{k-1}}) \\ =& f(M_{t_{k-1}}) E((M_{t_k} - M_{t_{k-1}}) | \mathcal{F}_{t_{k-1}}), \end{aligned}$$

and the last term is obviously equal to zero as M is a martingale.

Local and true martingales

The following criterion is often useful to verify that a local martingale is a true martingale.

Proposition 36

Let M be a local martingale. Then the following two assertions are equivalent.

(i) *M is a true martingale and $E(M_t^2) < \infty \ \forall t \geq 0$.*

(ii) *$E([M]_t) < \infty \ \forall t$.*

If either (i) or (ii) holds, we have $E(M_t^2) = E([M]_t)$.

Example. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is bounded and $\mathcal{C}^1$ and B is Brownian motion. Then $I_t = \int_0^t f(B_s) dB_s$ is a true martingale.

Why price processes of infinite variation?

Proposition 37

Consider a local martingale M with continuous trajectories of finite variation Then the paths of M are constant, i.e. $M_t = M_0$ almost surely.

Comment Note that the result does not hold for martingales with jumps as is shown by the compensated Poisson process.

Proof By Itô's-formula we get for M_t^2

$$M_t^2 = M_0^2 + 2\int_0^t M_s dM_s + [M]_t = M_0^2 + 2\int_0^t M_s dM_s\,,$$

as $[M]_t = 0$. Hence M_t^2 is a local martingale. Assume for simplicity that both M_t and M_t^2 are a real martingales. Then

$$0 \leq E\left((M_t - M_0)^2\right) = E\left(M_t^2 - 2M_t M_0 + M_0^2\right) = M_0^2 - 2M_0^2 + M_0^2 = 0\,,$$

which shows that $E\,(M_t - M_0)^2 = 0$ so that $M_t = M_0$ a.s.

Covariation

Fix a sequence τ_n of partitions of $[0, \overline{T}]$ with $\tau_n \to 0$ and continuous functions X, Y which admit a continuous quadratic variation $[X]_t$ and$[Y]_t$ along the sequence τ_n.

Definition Assume that for all $t \in [0, T]$ the following limit exists:

$$\lim_{n\to\infty} \sum_{t_i \in \tau_n;\, t_i \leq t} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}) =: [X, Y]_t.$$

Then $[X, Y]_t$ is called covariation of X and Y.

Theorem 38

$[X, Y]_t$ exists if and only if $[X + Y]_t$ exists; in that case we have the following so-called polarization-identity

$$[X, Y]_t = \frac{1}{2}\left([X + Y]_t - [X]_t - [Y]_t\right) . \tag{29}$$

Covariation: Examples

1) If X is a continuous function with continuous quadratic variation $[X]_t$ and A a continuous function of finite variation we have $[X+A]_t = [X]_t$ and hence $[X,A]_t = 0$.

2) Consider two independent Brownian motions B^1, B^2. Then $[B^1\cdot(\omega), B^2\cdot(\omega)]_t = 0$. To prove this claim we have to compute $[B^1+B^2]_t$. Note that $(B^1_t + B^2_t)/\sqrt{2}$ is again a Brownian motion and has therefore quadratic variation equal to t. Hence

$$\frac{1}{2}([B^1+B^2]_t - [B^1]_t - [B^2]_t) = \frac{1}{2}(2t - t - t) = 0\,.$$

3) Consider a continuous function X with continuous quadratic variation, and $\mathcal{C}^1$-functions f and g. Define $Y_t := \int_0^t f(X_s)dX_s$ and $Z_t := \int_0^t g(X_s)dX_s$. Then $[Y,Z]_t = \int_0^t f(X_s)g(X_s)d[X]_s$.

The d-dimensional Itô-formula

Theorem 39

Given continuous functions $X = (X^1, \ldots, X^d) : [0, \overline{T}] \to \mathbb{R}$ *with continuous covariation*

$$[X^k, X^l]_t = \begin{cases} [X^k]_t,\ k{=}l, \\ 1/2\left([X^k + X^l]_t - [X^k]_t - [X^l]_t\right),\ \ k \neq l \end{cases}$$

and a twice continuously differentiable function $F : \mathbb{R}^d \to \mathbb{R}$*. Then*

$$F(X_t) = F(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} F(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} F(X_s) d[X^i, X^j]_s\,.$$

d-dimensional Ito formula ctd

Remark on notation. For $\frac{\partial}{\partial x_i} F$ we often write F_{x_i}, $\frac{\partial^2}{\partial x_i \partial x_j} F$ is denoted by F_{x_i, x_j}. In short-notation the d-dimensional Itô-formula hence writes as:

$$dF(X_t) = \sum_{i=1}^{d} F_{x_i}(X_t) dX_t^i + \frac{1}{2} \sum_{i,j=1}^{d} F_{x_i, x_j}(X_t) d[X^i, X^j]_t \,.$$

Example. Let $W = (W^1, \ldots, W^d)$ be d-dimensional Brownian motion so that that $[W_t^k, W_t^l]_t = \delta_{kl} t$ where $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ otherwise. Hence we have

$$F(W_t) = F(W_0) + \sum_{i=0}^{d} \int_0^t F_{x_i}(W_s) dW_s^i + \frac{1}{2} \sum_{i=0}^{d} \int_0^t F_{x_i, x_i}(W_s) ds \,. \quad (30)$$

Implications of d-dimensional Ito

Corollary 40 (Itô's product formula)

Given X, Y with continuous quadratic variation $[X]_t$, $[Y]_t$ and covariation $[X, Y]_t$. Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t.$$

Short notation: $d(XY)_t = X_t dY_t + Y_t dX_t + d[X, Y]_t.$

Corollary 41 (Itô-formula for time-dependent functions)

Given a continuous function X with continuous quadratic variation $[X]_t$ and a function $F(t, x)$ which is $\mathbf{C}^1$ in t and $\mathbf{C}^2$ in x. Then

$$F(t, X_t) = F(0, X_0) + \int_0^t F_t(\cdot) ds + \int_0^t F_x(\cdot) dX_s + \frac{1}{2} \int_0^t F_{xx}(\cdot) d[X]_s.$$

Applications

1) *Geometric Brownian motion:* Given a Brownian motion W, an initial value $S_0 > 0$ and constants μ, σ with $\sigma > 0$ we define geometric Brownian motion S by

$$S_t = S_0 \exp\left(\sigma W_t + (\mu - 1/2\sigma^2)t\right) .$$

Using Ito it can be shown that S_t satisfies the SDE

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dW_s$$

Itô processes.

A one-dimensional Itô process is the solution of a stochastic differential equations (SDE) driven by Brownian motion.

Definition 42

Given a Brownian motion $W_{t\geq 0}$, a time point $t_0 \geq 0$, some $x \in \mathbb{R}$ and functions $\mu : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, the process $X = (X)_{t\geq t_0}$ is called an Itô process with initial value t_0, x, drift μ and dispersion σ if X satisfies the SDE

$$X_t = x + \int_{t_0}^{t} \mu(s, X_s)\, ds + \int_{t_0}^{t} \sigma(s, X_s)\, dW_s\,. \tag{31}$$

Comments

- In short notation (31) reads $dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$.
- Euler approximation. For a small time step Δt one has with $t_n = n\Delta t$

$$X_{t_{n+1}} = X_{t_n} + \mu(t_n, X_{t_n})\Delta t + \sigma(t_n, X_{t_n})\epsilon_n$$

  where ϵ_n are iid $\sim N(0, \Delta t)$. This is the standard approach for simulation of X.
- As before the stochastic integral $I_t = \int_{t_0}^{t} \sigma(s, X_s)\,dW_s$ is defined as limit of non-anticipating sums.

$$I_t = \lim_{n\to\infty} \sum_{t_i \in \tau_n, t_i \le t} \sigma(t_{i-1}, X_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})$$

- Many results on existence and uniqueness in literature

Examples

In the following we let for simplicity $t_0 = 0$.

- Arithmetic BM, $\mu(t, X) = \alpha$, $\sigma(t, x) = \beta$, $\Rightarrow$
  $X_t = x + \alpha t + \beta W_t \sim N(x + \alpha t, \beta^2 t)$.
- Geometric BM, $\mu(t, X) = \alpha x$, $\sigma(t, x) = \beta x$, $x > 0$

$$\Rightarrow X_t = x \exp\left((\alpha - \frac{1}{2}\beta^2)t + \beta W_t\right)$$

- Ornstein Uhlenbeck process. $dX_t = \kappa(\theta - X_t)dt + \beta dW_t$, $X_0 = \bar{r}$ for $\beta > 0$. Here $\mu(t, X) = \kappa(\theta - x)$, $\sigma(t, x) = \beta$.

  Exercise: solution is given by

$$X_t = e^{-\kappa t}\bar{r} + \theta(1 - e^{-\kappa t}) + \beta e^{-\kappa t} \int_0^t e^{\kappa s} dW_s$$

Mathematical properties.

- The quadratic variation of an Ito process equals $[X]_t = [I]_t = \int_{t_0}^t \sigma^2(s, X_s)ds$.
- Ito processes are Markov processes; for $T > t$ it holds that

$$E(f(X_T) \mid \mathcal{F}_t) = E(f(X_T) \mid X_t).$$

  If μ and σ do not depend on time, that is $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ we get

$$E\big(f(X_T) \mid \mathcal{F}_t\big)(\omega) = E_{X_t(\omega)}\big(f(X_{T-t})\big)$$

  where $E_x(\cdot)$ describes expectation with respect to the distribution of the solution X with $X_0 = x$.
- Ito processes are semimartingales with

$$M_t = \int_0^t \sigma(s, X_s)dW_s \text{ and } A_t = \int_0^t \mu(s, X_s)ds\,.$$

Ito formula for Ito processes.

If X is an Ito process the Ito formula takes a special form

Theorem 43

Consider an Ito process with dynamics $dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$ *and a function* $F\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ *which is* $\mathbf{C}^1$ *in* t *and* $\mathbf{C}^2$ *in* x. *Then*

$$F(t, X_t) = F(0, X_0) + \int_0^t F_t(\cdot) ds + \int_0^t F_x(\cdot)\mu(s, X_s) ds + \int_0^t F_x(\cdot)\sigma(s, X_s) dW_s + \frac{1}{2}\int_0^t F_{xx}(\cdot)\sigma^2(s, X_s) ds.$$

Feynman Kac formula

Consider functions $\mu(x)$, $\sigma(x)$, $r(x)$ and some function ϕ on $\mathbb{R}$. Suppose that $F(t,x)$ solves the terminal value problem

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \mu(x)\frac{\partial F}{\partial x}(x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 F}{\partial x^2}(x) =& r(x)F(t,x), \qquad (32) \\ F(T,x) =& \phi(x). \qquad (33) \end{aligned}$$

Goal: represent F in terms of some Itô process.

Consider the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_{t_0} = x. \qquad (34)$$

Then we have the following result

One-dimensional Feynman Kac formula II

Theorem 44 (Feynman Kac)

If F is sufficiently integrable (eg. bounded), it holds for $t_0 \leq T$ that

$$F(t_0, X_{t_0}) = E_{X_{t_0}}\left(\exp\left(-\int_0^{T-t_0} r(X_s)ds\right)\phi(X_{T-t_0})\right). \qquad (35)$$

(35) is called Feynman Kac formula. It can be used in two ways:

- We can use probabilistic techniques or Monte-Carlo simulation to compute the expectation (35) in order to solve numerically the PDE (32), (33).
- We can solve the PDE (32), (33), perhaps numerically, in order to compute the expectation (35).

For an extension to the time dependent case and further generalisations we refer to the literature such as [Karatzas and Shreve, 1988]

Overview

The Black Scholes Model

Assets. We consider a market with two assets.

- money market account B with $B_t = \exp(rt)$ for some $r > 0$
- stock price S risky.

Stock price dynamics. We work on filtered probability space $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}$ supporting a standard Brownian motion W_t. Possible models for the dynamics of S

- arithmetic Brownian motion (Bachelier). $S_t = S_0 + \sigma W_t + \mu t$ for constants $\mu, \sigma > 0$. Problem: $S_t \sim N(S_0 + \mu t, \sigma^2 t)$, so that we get negative stock prices.
- geometric Brownian motion ([Samuelson, 1965])

$$S_t = S_0 \exp\left(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t\right). \tag{36}$$

Geometric Brownian motion, also known as Black-Scholes model, is popular until today.

Properties of Black-Scholes model

1) S solves the linear stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t \,, \quad S_0 = S_0. \tag{37}$$

Intuitive description: $\frac{S_{t+h}-S_t}{S_t} \approx \mu h + \sigma\varepsilon$ for $\varepsilon \sim N(0,h)$.
2. Formula for S implies that log-returns

$$\ln S_{t+h} - \ln S_t = \sigma(W_{t+h} - W_t) + (\mu - \frac{1}{2}\sigma^2)h$$

are $N((\mu - \frac{1}{2}\sigma^2)h, \sigma^2 h)$-distributed; in particular the volatility σ is the instantaneous variance of the log-returns. Moreover, under (36) non-overlapping log-returns are stochastically independent.

Critical discussion of geometric Brownian motion

Advantages

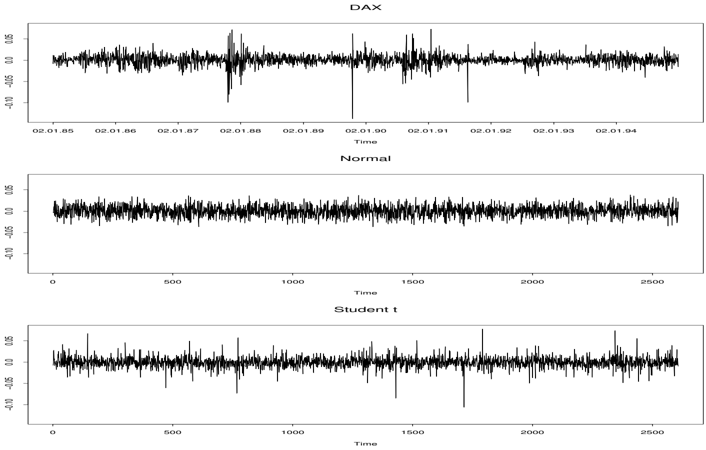
- Geometric Brownian motion gives a reasonable first description for asset prices data but the model is far from perfect.
- Geometric Brownian motion allows for explicit pricing formulae for a relatively large class of derivatives.
- The Black-Scholes model is quite robust as a model for hedging of derivatives: if real asset-price dynamics are 'not too different from geometric Brownian motion' hedging strategies computed using the Black-Scholes model perform reasonably well.

Black Scholes and financial time series

Typical financial data show strong evidence against the assumption of iid normally distributed log returns. If (S_t) denotes (say daily) values of an asset price and returns (X_t) are defined by $X_t = \ln(S_t/S_{t-1})$, we generally observe a number of stylized facts:

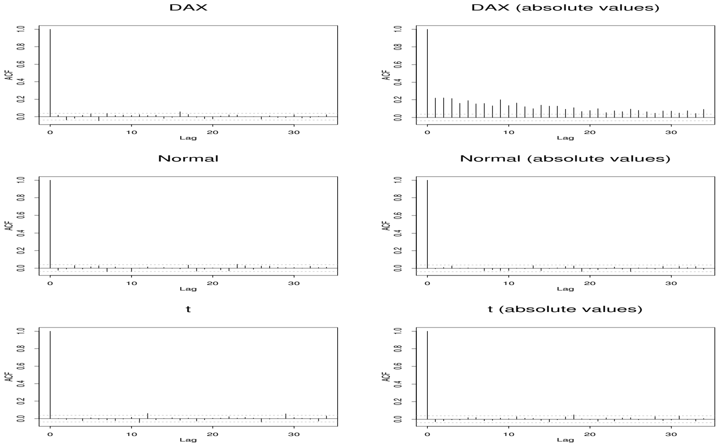
- Returns not iid but correlation low
- Absolute returns highly correlated
- Volatility appears to change randomly with time
- Returns are leptokurtic or heavy–tailed

Stylized Facts: Volatility



Log-returns for DAX index from 02.01.85 until 30.12.94 compared with simulated iid data from fitted normal and t distribution.

Stylized Facts: Autocorrelation



Correlograms for the three datasets in previous figure.

Pricing of terminal-value claims

We consider pricing of terminal value claims with payoff $H = h(S_T)$.
(more general claims require more general Ito integral)

Basic notions

- A Markov trading strategy is given by a pair of smooth functions $(\phi(t,S),\ \eta(t,S))$, where $\phi(t,S_t)$ and $\eta(t,S_t)$ give the number of stocks / units of the money market account in the portfolio at time t.
- The value at time t of the strategy is $V(t,S_t) = S_t\phi(t,S_t) + \eta(t,S_t)B(t)$. Note that

  $$\eta(t,S) := (V(t,S) - S\phi(t,S))\,/B(t)$$

  so that portfolio can be described via V and ϕ

Selffinancing strategies and gains from trade

Motivation Given a Markov trading strategy (ϕ, η). and a sequence τ_n of partitions with $|\tau_n| \to 0$. Define piecewise constant approximations to (ϕ, η) by

$$\phi^n_t(\omega) = \sum_{t_i \in \tau_n} \phi(t_{i-1}, S_{t_{i-1}}(\omega))\, 1_{(t_{i-1}, t_i]}(t) \tag{38}$$

$$\eta^n_t(\omega) = \sum_{t_i \in \tau_n} \eta(t_{i-1}, S_{t_{i-1}}(\omega))\, 1_{(t_{i-1}, t_i]}(t) \tag{39}$$

and let $V^n_t = \phi^n_t S_t + \eta^n_t B_t$. Discrete-time finance $\Rightarrow$ piecewise constant strategy is selffinancing $\Leftrightarrow$ for all $t_i \in \tau_n$

$$V^n_{t_i} = V_0 + G^n_{t_i}, \text{ where } G^n_{t_i} = \sum_{j=1}^{i} \left(\phi^n_{t_j}(S_{t_j} - S_{t_{j-1}}) + \eta^n_{t_j}(B_{t_j} - B_{t_{j-1}}) \right) .$$

Selffinancing strategies and gains from trade ctd

By definition of the Itô-integral, G_t^n converges to $\int_0^t \phi(s, S_s)dS_s + \int_0^t \eta(s, S_s)dB_s$. Hence the following definition is natural.

Definition 45

Given a Markov trading strategy $(\phi(t, S_t^1), \eta(t, S_t^1))$ induced by smooth functions $\phi,\ \eta : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$.

(i) The gains from trade of this strategy are given by

$$G_t = \int_0^t \phi(s, S_s)dS_s + \int_0^t \eta(s, S_s)dB_s\,.$$

(ii) The strategy is selffinancing, if $V(t, S_t) = V(0, S_0) + G_t$ for all $t \leq T$.

Note that gains from trade can be written as
$G_t = \int_0^t \phi(s, S_s)dS_s + \int_0^t r\eta(s, S_s)B_s)ds$

Pricing and Hedging Terminal-Value Claims

Definition. Consider a terminal value claim with payoff $h(S_T)$. A selffinancing strategy is a replicating strategy for the claim if $V(T,S) = h(S)$ for all $S > 0$; in that case $V(t, S_t)$ is the fair price of the claim at time t.

Theorem 46

1) Let $V : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ be a continuous function which solves the PDE

$$V_t(t,S) + \frac{1}{2}\sigma^2 S^2 V_{SS}(t,S) + rSV_S(t,S) = rV(t,S)\,, (t,S) \in [0,T) \times \mathbb{R}^+\,. \tag{40}$$

Then the hedging strategy with stock-position $\phi(t,S) = V_S(t,S)$ and value $V(t,S)$ is selffinancing.

2) If V satisfies $V(T,S) = h(S)$, the strategy replicates the terminal value claim with payoff $h(S_T)$ and the fair price at time t equals $V(t,S_t)$.

Proof

1. Quadratic variation of GBM. Recall that

$$S_t = S_0 + \int_0^t \sigma S_s dW_s + \int_0^t \mu S_s ds =: M_t + A_t \,.$$

As A is of finite variation we get $[S]_t = [M]_t = \int_0^t \sigma^2 (S_s)^2 ds$.

2. The rest of the proof is an application of the Ito formula.

$$
\begin{aligned}
V(t, S_t) &= V(0, S_0) + \int_0^t V_S dS_s + \int_0^t V_t ds + \frac{1}{2}\int_0^t V_{SS} d[S]_s \\
&= V(0, S_0) + \int_0^t V_S dS_s + \int_0^t \left(V_t + \frac{1}{2}\sigma^2 (S_s)^2 V_{SS} \right) ds \,.
\end{aligned}
$$

Using the PDE (40) and the definition of ϕ this equals

$$
\begin{aligned}
&= V(0, S_0) + \int_0^t \phi(s, S_s) dS_s + \int_0^t r(V - \phi\, S_s) ds \\
&= V(0, S_0) + \int_0^t \phi(s, S_s) dS_s + \int_0^t \eta(s, S_s) dB_s,
\end{aligned}
$$

where $\eta(t, S_t) = (V(t, S_t) - \phi(t, S_t) S_t)/B(t)$

The second claim is obvious.

Risk neutral pricing formula via Feynman Kac

Recall that fair value $V_t = V(t, S_t)$ of a terminal-value claim with payoff $h(S_T)$ in the Black-Scholes Model solves the terminal value problem

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV, \quad V(T, S) = h(S). \tag{41}$$

Using Feynman Kac we get a risk-neutral pricing formula from (41):

Theorem 47 (Risk neutral pricing)

It holds that

$$V(t, S) = E_S\big(e^{-r(T-t)} h(S_{T-t})\big), \quad t \leq T. \tag{42}$$

where S solves the SDE $dS_t = rS_t dt + \sigma S_t dW_t$.

This is a risk-neutral pricing formula in continuous time. In particular, the stock price has drift r (instead of μ) so that the discounted price process $e^{-rt} S_t$ is a martingale.

The Black-Scholes Formula

To price a European call option we have two possibilities:

- Solve the PDE (40) with $h(S) = (S - K)^+$. This is done by reduction to heat equation.
- Compute the risk neutral expectation (42)

Approach via heat equation

Lemma. Define $\tau(t) = \sigma^2(T-t)$ and $z(t,S) = \ln S + (r - \frac{1}{2}\sigma^2)(T-t)$. Denote by $u(t,z): [0,T] \times \mathbb{R} \to \mathbb{R}$ the solution of the heat-equation $u_t = \frac{1}{2}u_{zz}$ with initial condition $u(0,z) = (e^z - K)^+$. Then

$$C(t,S) := e^{-r(T-t)} u(\tau(t), z(t,S))$$

solves the terminal value problem for the price of a European call.

Proof. We have $C(T,S) = u(\tau(T), z(T,S)) = u(0, \ln S) = (S-K)^+$, so that the function C has the right value at maturity. Moreover,

$$\frac{\partial C}{\partial t} = e^{-r(T-t)} \left(ru - \sigma^2 u_\tau + (1/2\sigma^2 - r) u_z \right)$$

$$\frac{\partial C}{\partial S} = e^{-r(T-t)} u_z 1/S \,, \; \frac{\partial^2 C}{\partial S^2} = e^{-r(T-t)} \left(u_{zz}(1/S)^2 - u_z (1/S)^2 \right)$$

Plugging these expressions into the PDE (40) yields the result.

Solving the heat equation ctd

It is well-known that the solution u of the heat-equation with initial condition $u(0,z) = u_0(z)$ equals

$$u(\tau, z) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} u_0(x) e^{-\frac{(z-x)^2}{2\tau}} dx.$$

This gives after tedious computations

Theorem 48 (Black Scholes formula)

The price of a European call with strike K and maturity date T in the Black-Scholes model with volatility σ and interest rate r equals

$$C_{BS}(t, S; \sigma, r, K, T) := SN(d_1) - e^{-r(T-t)} KN(d_2)\,, \text{ with} \tag{43}$$

$$d_1 = \frac{\ln S/K + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t}\,.$$

Black Scholes formula: probabilistic approach

We know from Feynman Kac that $C_t = e^{-r(T-t)} E\big((S_{T-t} - K)^+\big)$ where

$$S_{T-t} = S_0 \exp\big(\sigma W_{T-t} + (r - \frac{1}{2}\sigma^2)(T-t)\big) = \exp(Z),$$

and $Z \sim N\big(\ln S_0 + (r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)\big)$. This expectation can be computed explicitly.

Call Price ctd

For simplicity $t = 0$. Goal: compute $E\big((e^Z - K)^+\big)$ for $Z \sim N\big(\ln S_0 + (r - \frac{1}{2}\sigma^2)(T), \sigma^2(T)\big)$. Let

$$\alpha := \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 T}}, \quad \mu := \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T, \quad \tilde{\sigma} := \sigma\sqrt{T}.$$

This gives

$$e^{-rT} E\Big(\big(e^{Z_T} - K\big)^+\Big) = \alpha \int_{-\infty}^{\infty} \left(e^x - K\right)^+ \exp\left(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2}\right) dx$$

$$= \underbrace{\alpha \int_{\ln K}^{\infty} e^x \exp\left(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2}\right) dx}_{=:I_1} - K \underbrace{\alpha \int_{\ln K}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\tilde{\sigma}^2}\right) dx}_{=:I_2}.$$

We concentrate on I_1.

Computing I_1

Approach: write integrand as normal density with new mean and deterministic correction term. Integrand is of the form $\exp\Big(\lambda(x)\Big)$ with

$$
\begin{aligned}
\lambda(x) &= x - \frac{(x-\mu)^2}{2\tilde{\sigma}^2} = -\frac{-2\tilde{\sigma}^2 x + x^2 - 2\mu x + \mu^2}{2\tilde{\sigma}^2} \\
&= -\frac{\Big(x - (\mu + \tilde{\sigma}^2)\Big)^2 + \Big(\mu^2 - (\mu+\tilde{\sigma}^2)^2\Big)}{2\tilde{\sigma}^2} \\
&= -\frac{\Big(x - (\ln S_0 + (r + \frac{\sigma^2}{2})T)\Big)^2}{2\sigma^2 T} + (\ln S_0 + rT),
\end{aligned}
$$

where last equality follows from $\mu + \tilde{\sigma}^2 = \ln S_0 + (r + \frac{\sigma^2}{2})T$ and $\frac{(\mu+\tilde{\sigma}^2)^2-\mu^2}{2\tilde{\sigma}^2} = \mu + \frac{\tilde{\sigma}^2}{2} = \ln S_0 + rT$.

Using $\alpha e^{\ln S_0 + rT} = \frac{1}{\sqrt{2\pi\sigma^2 T}} S_0$ we get

$$\alpha I_1 = \frac{S_0}{\sqrt{2\pi\sigma^2 T}} \int_{\ln K}^{\infty} \exp\Big(- \frac{\big(x - (\ln S_0 + (r + \frac{\sigma^2}{2})T)\big)^2}{2\sigma^2 T}\Big) dx.$$

Consider $\tilde{Z} \sim N\big(\ln S_0 + (r + \frac{\sigma^2}{2})T, \sigma^2 T\big)$. $\Rightarrow \frac{\tilde{Z} - \ln S_0 - (r+\sigma^2/2)T}{\sigma\sqrt{T}} \sim N(0,1)$ and we get

$$\begin{aligned} \alpha I_1 &= S_0 \, P(\tilde{Z} > \ln K) \\ &= S_0 \, P\Big(\frac{\tilde{Z} - \ln S_0 - (r + \sigma^2/2)T}{\sigma\sqrt{T}} > -d_1 \Big) \\ &= S_0 \Big(1 - N(-d_1) \Big) \; = \; S_0 N(d_1), \end{aligned}$$

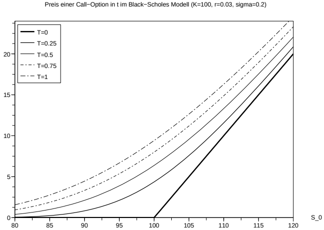
Similarly, $\alpha I_2 = e^{-rT} K N(d_2)$; for this term no quadratic substitution necessary.

Put options.

Recall put-call parity for price C_t and P_t of European call and put:
$C_t + e^{-r(T-t)}K = S_t + P_t$.
This gives for price P_t of a European put

$$P_t = -S_t N(-d_1) + Ke^{-r(T-t)}N(-d_2)\,.$$

Call and Put price



Price of a call (left) and of a put (right) for varying time to maturity

Delta of an option

The delta of an option is the derivative wrt the price of the underlying. In the Black Scholes model we have

$$\Delta_C = \frac{\partial C}{\partial S} = N(d_1) \text{ and } \Delta_P = \frac{\partial P}{\partial S} = \Delta_C - 1 = -N(-d_1)$$

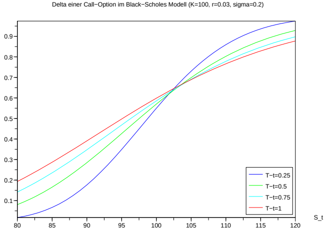
Hedging. The Delta is relevant for so-called delta-hedging.

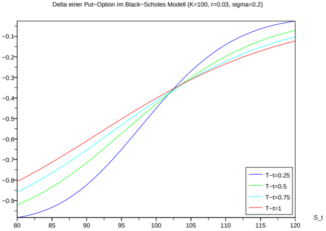
- The hedge-portfolio for a call consists of $\frac{\partial}{\partial S} C_{BS} = N(d_1)$ units of S and $(C_{BS}(t,S) - N(d_1)S)/e^{rt} = -e^{-rT}KN(d_2)$ units of B.
- The hedge-portfolio for a put consists of $\frac{\partial}{\partial S} P_{BS} = -N(-d_1)$ units of S and of

$$(P_{BS}(t,S) + (1 - N(d_1))S)/e^{rt} = e^{-rT}KN(-d_2)$$

units of B.

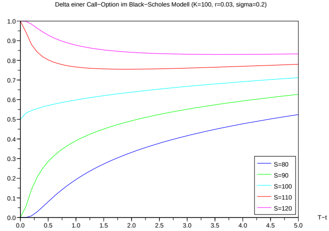
Delta of an option

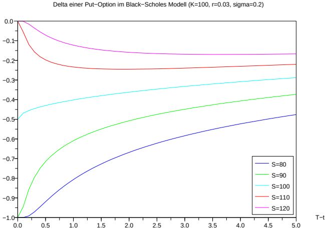




Delta of a call (left) and of a put (right) as a function of S for different time to maturity; throughout $K = 100$.

Delta of an option

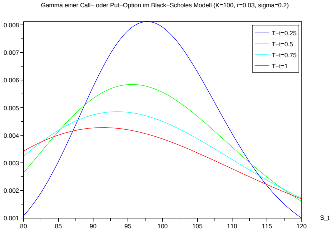


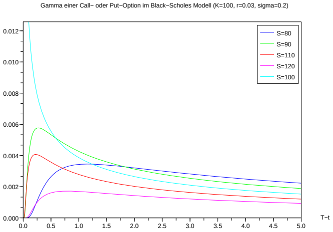


Delta of a call (left) and of a put (right) as a function of time to maturity $T - t$ for different S

Gamma

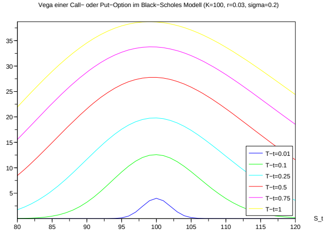
The Gamma is the second derivative wrt the underlying: $\Gamma_C = \frac{\partial^2 C}{\partial S^2}$. It holds that $\Gamma_C = \Gamma_P = \frac{\varphi(d_1)}{S_t \sigma \sqrt{T-t}}$, φ density of $N(0,1)$. Gamma measures how fast Delta changes and hence how often a hedge needs rebalancing.

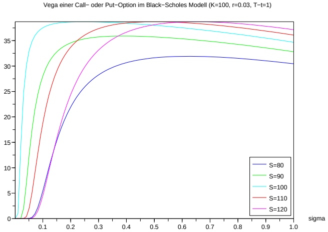




Vega

The Vega (not really a Greek letter) is the derivative wrt volatility: $\text{Vega}_C = \frac{\partial C}{\partial \sigma}$. It holds that $\text{Vega}_C = \text{Vega}_P = S_t \varphi(d_1)\sqrt{T-t}$. Vega is always positive.



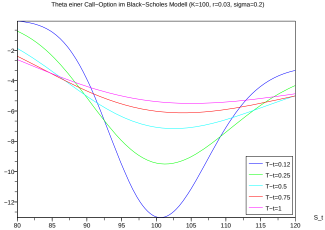


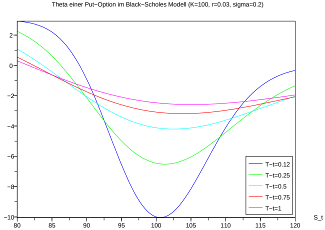
Further Greeks

The other partial derivatives have names as well.

$$\Theta_C = \frac{\partial C}{\partial t} = -\frac{S_t \sigma \varphi(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2) \text{ Theta (time sensitivity)}$$
$$\Theta_P = \frac{\partial P}{\partial t} = -\frac{S_t \sigma \varphi(d_1)}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$$
$$\rho_C = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2) \text{ Rho (interest-rate sensitivity)}$$
$$\rho_P = \frac{\partial P}{\partial r} = -K(T-t)e^{-r(T-t)}N(-d_2)$$

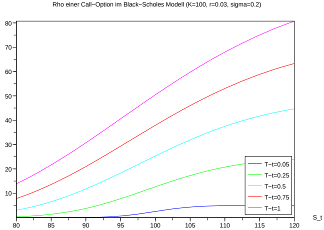
Theta of a call and a put

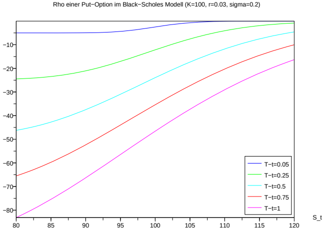




Theta of a call (left) and a put (right) as a function of S for different $T - t$

Rho of a call and a put





Rho of a call (left) and a put (right) as a function of S for different $T - t$

Hedging

Suppose that a trader has sold an option C and that wants to hedge the risk.

Theory. A continuously rebalanced selffinancing portfolio consisting of $\Delta_t = C_S^{\mathrm{BS}}(t, S_t)$ shares and an initial value $V_0 = C^{\mathrm{BS}}(0, S_0)$ provides a perfect hedge.

Practice. Only discrete rebalancing possible; markets do not follow Black Scholes. Hence there is a hedging error. It depends on

- rebalancing frequency
- structure (nonlinearity) of C
- 'true' asset price dynamics

In the sequel we provide a mathematical result making this precise.

Model Risk in the Black-Scholes Model

We now study the implications of volatility mis-specification and stochastic volatility for the performance of hedging strategies. We assume that the asset price follows the SDE

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t$$

for some – possibly stochastic – volatility σ_t. For simplicity we assume that $r = 0$.

Consider a trader who uses the Black-Scholes model with volatility σ^* in pricing and hedging a terminal value claim. Denote by h^{BS} the solution of the PDE terminal value problem for the BS model with $r = 0$, i.e.

$$h_t^{BS}(t, S) + \frac{1}{2}(\sigma^*)^2 S^2 h_{SS}^{BS} = 0 \,, \quad h^{BS}(T, S) = h(S) \,. \tag{44}$$

The tracking error

Assume that the trader holds $h_S^{BS}(t, S_t)$ shares of stock at time t and that he maintains a selffinancing portfolio. Then the actual value at T of his portfolio equals $V_T = V_0 + \int_0^T h_S^{BS}(t, S_t)dS_t$.

Definition 49

The tracking error of the hedge is given by $e_T = h(S_T) - V_T$.

Note that the hedge produces a loss if $e_T > 0$ and a gain if $e_T < 0$.

Proposition 50 ([El Karoui et al., 1998])

$$e_T = \frac{1}{2}\int_0^T (S_t)^2 \left(\sigma_t^2 - (\sigma^*)^2\right) h_{SS}^{BS}(t, S_t)dt.$$

Comments

- The tracking error is proportional to $(\sigma_t^2 - (\sigma^*)^2)$, the estimation error for volatility, and to the average of the 'Gamma' $h_{SS}^{BS}(t, S_t)$ over the future path of the stock-price process.
- If $h_{SS}^{BS}(t, S_t) > 0$ the hedge looses (gains) money if $\sigma_t > \sigma^*$ $(\sigma_t < \sigma^*)$;
- If $h_{SS}^{BS}(t, S_t) < 0$ the hedge looses (gains) money if $\sigma_t < \sigma^*$ $(\sigma_t > \sigma^*)$.

Proof

As $h^{BS}(T,S) = h(S)$ we get from Itô's formula:

$$h(S_T) = h^{BS}(0,S_0) + \int_0^T h_S^{BS} dS_t + \int_0^T \left(h_t^{BS} + \frac{1}{2}\sigma_t^2 (S_t)^2 h_{SS}^{BS} \right) dt \,.$$

This implies that

$$e_T = \int_0^T \left(h_t^{BS} + \frac{1}{2}\sigma_t^2 (S_t)^2 h_{SS}^{BS} \right) dt \,.$$

The Black-Scholes PDE gives $h_t^{BS}(t,S) = -\frac{1}{2}(\sigma^*)^2 S^2 h_{SS}^{BS}(t,S)$; hence

$$e_T = \frac{1}{2} \int_0^T (S_t)^2 (\sigma_t^2 - (\sigma^*)^2) h_{SS}^{BS}(t,S_t) dt \,.$$

Volatility Estimation

Two common approaches for determination of σ.

Historical volatility: (statistical considerations.) Under (37) log-returns over periods of length Δ are independent and $N((\mu - \frac{1}{2}\sigma^2)\Delta, \sigma^2\Delta)$ distributed. Given asset price data at times t_i, $i = 1, \dots, N$ with $t_i - t_{i-1} = \Delta$ (e.g. daily returns) define $Y_i = \ln S_{t_i} - \ln S_{t_{i-1}}$. The standard estimator for σ_Δ, is given by

$$\hat{\sigma}_\Delta = \left(\frac{1}{N-1} \sum_{i=1}^{N} \left(Y_i - \overline{Y} \right)^2 \right)^{\frac{1}{2}}, \quad \text{where } \overline{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i.$$

Estimated historical volatility is then given by $\hat{\sigma}_{hist} = \hat{\sigma}_\Delta / \sqrt{\Delta}$ (as one year has $1/\Delta$ periods).

2. Implied volatility.

Idea: use market estimate of σ that is embedded in traded options to price non-traded derivatives. Consider the following example:
Assume that a call K and T is traded at time t and at a given stock-price $(S_t)^*$ for a price of C_t^*. The implied volatility $\hat{\sigma}_{impl}$ is then given by the solution to

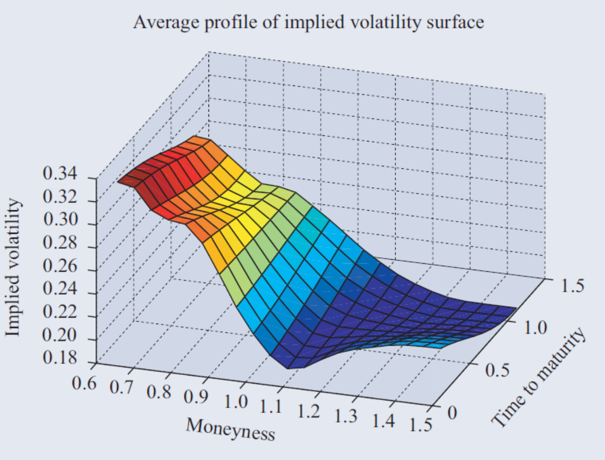
$$C_{BS}(t, (S_t)^*; \hat{\sigma}_{impl}, K, T) = C_t^* .$$

As C_{BS} is strictly increasing in σ a unique solution to this equation exists; it is usually determined by numerical procedures.

Smiles and Skews of implied volatility

- If market 'believes' in Black-Scholes model implied volatilities should be constant, independent of K and $T - t$.
- In reality implied volatilities display a typical pattern known as implied volatility skew (or smile)
- Skew represents deviations of reality from Black Scholes (volatility clustering, heavy tails, jumps etc)
- Searching for models that can explain volatility skew was (and still is) is an important research area in mathematical finance.

Smiles and Skews of implied volatility



A typical volatility skew (taken from Cont-Fonseca 2001); moneyness is defined as K/S_t.

Currency Options

Setup. Two countries, foreign (USD) and domestic (EUR). Domestic and foreign interest rate is r^d resp. r^f; savings accounts given by $B_t^d = \exp(r^d t)$ and $B_t^f = \exp(r^f t)$ (B^f quoted in USD). Exchange rate e_t satisfies

$$de_t = \mu e_t dt + \sigma e_t dW_t \text{ with initial value } e_0$$

Goal. Price a European call with payoff $(e_T - K)^+$.

Selffinancing strategies. Consider replicating portfolio with $\phi(t, e_t)$ USD and cash position $\eta(t, e_t)$; value $V_t = V(t, e_t)$. Investing in USD we get an interest income over $(t, t+h]$ of approx. $r^f \phi(t, e_t) e_t h$. Hence portfolio selffinancing if

$$dV(t, e_t) = \phi(t, e_t) de_t + \eta(t, e_t) r^d dt + r^f \phi(t, e_t) e_t dt\,.$$

Currency Options ctd.

A PDE for the price. Ito formula gives

$$dV(t,e_t) = (V_t + \frac{1}{2}\sigma^2 e_t^2 V_{ee})(t,e_t)dt + V_e(t,e_t)de_t$$

Comparing with selffinancing condition we get that $\phi(t,e) = V_e(t,e)$ and the PDE

$$V_t + (r^d - r^f)eV_e + \frac{1}{2}\sigma^2 e^2 V_{ee} = r^d V, \quad V(T,e) = (e-K)^+.$$

Risk-neutral dynamics of e_t. Feynman Kac gives

$$C(t,e) = E_e\left(e^{-r^d(T-t)}(e_{T-t} - K)^+\right) \text{ with } de_t = (r^d - r^f)e_t dt + \sigma e_t dW_t.$$

Garman Kohlhagen formula. Evaluating the expectation we get

$$C(t,e_t) = e_t e^{-r^f(T-t)} N(d_1) - Ke^{-r^d(T-t)} N(d_2),$$

for $d_1 = \frac{\ln(e_t) - \ln K + (r^d - r^f + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$.

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